Tests for Convergence of Series 1

Wednesday, April 12, 2023 8:47 AM

recall the MCT (monotone convergence theorem)

corollary of MCT: suppose (a_n) is non-negative, then $\sum_{n=1}^{\infty} a_n$ converges $\xleftrightarrow{\text{"if \& only if"}}$ partial sums $s_m = a_1 + \ldots + a_m$ bounded (above since $a_n \to 0$)

(corollary of theorem 6)

ex 1) $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$ converge?

since $a_n = \frac{1}{n^2}$ is non-negative, suffice (by thm) that s_m bounded

$$s_m = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{m^2} \le \frac{1}{1^2} + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \ldots + \frac{1}{m(m-1)}$$

$\frac{1}{3\cdot 3} \to 2$

$3\cdot 2 < 3\cdot 3$

make denom smaller, > fraction → > series

doesn't work with equation / keep 1

$$\left[\frac{A}{m} + \frac{B}{m-1} = \frac{1}{m(m-1)}\right] m(m-1)$$

$A(m-1) + Bm = 1$

$Am + Bm - A = 1$

$-A = 1$ $\quad (A+B) = 0$

$A = -1$ $\quad B = 1$

$\frac{-1}{m} + \frac{1}{m-1} = \frac{1}{m(m-1)}$

$$= \frac{-1}{m} + \frac{1}{m-1} = \frac{1}{m-1} - \frac{1}{m}$$

(telescoping theories)

$$= 1 + \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \ldots + \left(\cancel{\frac{1}{m-1}} - \frac{1}{m}\right) = 2 - \frac{1}{m} < 2$$

$\boxed{\text{converges}}$

$\overset{s_1}{1} \le s_m < \overset{s_\infty}{2}$ bounded → converges

(thm 9)

integral test: suppose $a_n = f(n)$ with $f: \mathbb{R} \xrightarrow{\text{continuous}} \mathbb{R}$ ≥ 0 (decreasing), then

not decreasing, diverge

"continuous sum" $\quad$ "discrete sum"

$$\int_1^{\infty} f(x)\,dx \overset{(\neq)}{\sim} \sum_{n=1}^{\infty} a_n \quad (a_n = f(n))$$

changed to 137 $\quad$ doesn't tell # $\quad$ (could start @ n = 137)

either both convergence or divergence

application of integral test: series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $(p = +\mathbb{R})$ = $\begin{cases} \text{converges} & \text{for } p > 1 \\ \text{diverges} & \text{for } p \le 1 \end{cases}$

ex 1) $p = 1$ $\quad \sum_{n=1}^{\infty} \frac{1}{n} \to f = \frac{1}{x}$

$p = \frac{1}{2}$ $\quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{3}} \to f = \frac{1}{\sqrt{x}}$

apply integral test to $\int_1^{\infty} \frac{1}{x^p}\,dx$

$\int \frac{1}{x} = \ln|x| \to \infty$

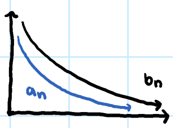
$\int \frac{1}{\sqrt{x}} = \int x^{-1/2} = 2x^{1/2} \to \infty$

$\int \frac{1}{x^2} = \int x^{-2} = -1x^{-1} = \frac{-1}{x} \to 0$

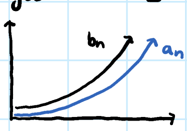
(thm 10)

direct comparison: $\sum a_n$ & $\sum b_n$ two series where $0 \le a_n \le b_n$ then ...

1) $\sum b_n$ converges $\longrightarrow$ $\sum a_n$ converges



2) $\sum a_n$ diverges $\longrightarrow$ $\sum b_n$ diverges



(thm 11)

limit comparison: suppose that a_n, b_n sequences $(a_n, b_n > 0)$ ...

- $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$ finite then $\sum a_n$ & $\sum b_n$ both converge or diverge (similar growth)
- $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ then $\underset{\text{converges}}{\sum a_n}$ if $\underset{\text{converges}}{\sum b_n}$ (grows faster)
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ then $\underset{\text{diverges}}{\sum a_n}$ if $\underset{\text{diverges}}{\sum b_n}$ (grows faster)

* choose b_n similar in rate of change as a_n, but simplier *

$\sum^{\infty} \frac{3^n - n}{\ldots}$ $\quad \sum^{\infty}$ $\quad \frac{3^n - n}{\ldots}$ $\quad \frac{3^n}{\ldots}$ $\quad \lim \ldots = \infty$

grows faster ... diverges diverges

ex 1) $\sum_{n=1}^{\infty} \frac{3^n - n}{n^2} = \sum_{n=1}^{\infty} a_n$

make $= a_n$

& then choose b_n

$a_n = \frac{3^n - n}{n^2}$ $\quad b_n = \frac{3^n}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3^n - n}{\cancel{n^2}} \cdot \frac{\cancel{n^2}}{3^n} = \lim_{n\to\infty} \frac{3^n - n}{3^n} = \lim_{n\to\infty} \frac{3^n}{3^n} - \frac{n}{3^n}$

$= 1 - 0 = 1$

$\lim_{n\to\infty} b_n = \infty$ $\quad \sum b_n$ converges, then $b_n \to 0$

b_n diverges, $\sum b_n$ diverges (theorem 6)

$\sum b_n$ = diverges

$\sum a_n$ = diverges

$\boxed{\sum_{n=1}^{\infty} \frac{3^n - n}{n^2} = \text{diverges}}$

ex 2) $\sum_{n=1}^{\infty} \frac{n^2 + 3n}{\sqrt[3]{n^7} - 7} = \sum_{n=1}^{\infty} a_n$

$a_n = \frac{n^2 + 3n}{n^{7/3} - 7}$ $\quad b_n = \frac{n^2}{n^{7/3}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2 + 3n}{n^{7/3} - 7} \cdot \frac{n^{7/3}}{n^2}$

$= \lim_{n\to\infty} \frac{n^{13/3} + 3n^{10/3}}{n^{13/3} - 7n^2}$

$= 1$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ = diverges

$b_n = \frac{n^2}{n^{7/3}}$

$= \frac{n^{6/3}}{n^{7/3}}$

$= n^{-1/3}$

$= \frac{1}{n^{1/3}}$

* p series *

$\sum a_n$ = diverge

$\boxed{\sum_{n=1}^{\infty} \frac{n^2 + 3n}{\sqrt[3]{n^7} - 7} = \text{diverge}}$